Static Models of Central Counterparty Risk

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February 27, 2014

Working Paper

Abstract

Following the 2009 G20 clearing mandate, international standard setting bodies (SSBs) have outlined a set of principles for CCP risk management; they have also devised CCP risk capital requirements on clearing members for their central counterparty exposures. There is still no consensus among CCP regulators, bank regulators, and CCPs on how central counterparty risk should be measured coherently in practice. A conceptually sound definition of the CCP risk capital in the absence of a unifying CCP risk measurement framework is challenging. Incoherent CCP risk capital requirements may create an obscure environment disincentivizing the central clearing of over the counter (OTC) derivatives transactions. Based on novel applications of well-known mathematical models in finance, this paper is the first to introduce a risk measurement framework that rigorously specifies all layers of the default waterfall resources of typical derivatives CCPs. The proposed framework can be used for a risk sensitive definition of the CCP risk capital based on which non-model-based less risk sensitive methods can be developed and evaluated.

Keywords: Risk Management, Clearing Mandate, Central Counterparty Risk, Risk Capital, Stochastic Models, Copulas, Monte Carlo Simulation

1 Introduction

The state of research on the optimal CCP design and the impact of derivatives CCPs on systemic risk is still not conclusive, and there is ongoing debate on whether central clearing through CCPs would make the financial system more stable (see, e.g., Pirrong [2013], Koepppl and Monnet [2013], Cont and Kokholm [2012], Duffie and Zhu [2011], and the references therein). Despite the uncertainties surrounding the CCPs’ optimal structure and their impact on the financial

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system, the 2009 G20 OTC derivatives reform program and the subsequent clearing mandate necessitate proper and practical central counterparty risk management.\footnote{In 2009, the G20 leaders agreed that all standardised OTC derivative contracts should be cleared through CCPs. The clearing mandate has been embodied in the Dodd-Frank Wall Street Reform and Consumer Protection Act and the European Market Infrastructure Regulation (EMIR).}

CCPs, which are intended to mitigate counterparty credit risk, have been widely employed in exchange-traded futures and options for decades. Consider a derivatives CCP that stands among a set of bilateral counterparties to OTC derivatives transactions – it becomes \textit{the seller to every buyer and the buyer to every seller}. Suppose that the CCP has \( n \) direct clearing members, \( CM_1, \ldots, CM_n \), and it holds a derivatives portfolio with each of them.\footnote{Indirect (client) clearing is not considered in this paper.} The financial resources of a CCP that are to protect it from the default of clearing members are often used in a pre-specified order and are commonly referred to as the \textit{default waterfall} resources of the CCP as summarized below. Upon the default of a clearing member with which the CCP has positive exposure, the defaulter resources are used first. These are the defaulter-pay variation margin, \( VM_i \), initial margin, \( IM_i \), and the prefunded default fund contribution of \( CM_i \), which is denoted by \( DF_i \). Variation and initial margin, which are defined in Section 2, depend on the value of the derivatives portfolio that the CCP holds with the clearing member over time and so are updated by the CCP on a frequent basis. Unlike variation and initial margins, the prefunded default funds are often defined qualitatively in practice and assumed as a priori-given quantities in the CCP research literature. The potential loss exceeding the defaulter-pay layer of the default waterfall is to be absorbed by the CCP itself and the surviving members’ resources. These are the CCP’s equity contributions, \( E \), and the survivor-pay prefunded default fund contributions. The potential loss exceeding these layers of the default waterfall is protected by the surviving members’ further contributions, which are referred to as unfunded default funds denoted by \( \tilde{DF} \), (see, e.g., Pirrong [2011] and Duffie et al. [2010] for more on default waterfall resources of typical derivatives CCPs).

Derivatives CCPs collectively take a relatively well defined and model-based approach to specify the variation and initial margins of the clearing members. However, the remaining layers of CCP default waterfall resources, i.e., the prefunded and unfunded default funds, are often specified qualitatively. This is mainly because the international standard setting bodies (SSBs) responsible for regulation of derivatives CCPs, outline broad and non-model-based principles for CCP risk management; particularly, for the default waterfall resources beyond initial margin. In contrast to the CCP regulatory risk management principles, SSBs have taken a formulaic approach to define the CCP risk capital of clearing members after the clearing mandate.\footnote{The committee on payment and settlement systems (CPSS) and the technical committee of the international organization of securities commissions (IOSCO) have developed regulatory principles for financial market infrastructures (FMIs), and the Basel committee on banking supervision (BCBS) has developed CCP risk capital charges in consultation with CPSS and IOSCO, (see CPSS and IOSCO [2012] and BCBS [2012]).} The clearing members’ regulatory CCP risk capital depends on all layers of default waterfall resources of a CCP in a complicated way, and so a coherent definition of the CCP risk capital in the absence of a model-based CCP risk measurement framework becomes quite difficult. In the absence of a unifying framework for default waterfall resources, derivatives CCP may engage
in considerably different risk management practices; the CCPs’ non-unifiable regulatory risk management framework could also create risk capital inconsistencies among CCPs and their clearing members.

This paper introduces a CCP risk measurement framework that specifies the default waterfall resources of derivatives CCPs rigorously by using well known mathematical modeling approaches in finance. More particularly, the proposed framework is based on an innovative application of the one-period portfolio credit risk models. The proposed framework can be used for comprehensive risk management of CCPs and for a risk sensitive definition of the CCP risk capital. This model-based risk sensitive approach to the CCP risk capital can be used to develop and evaluate non-model-based approaches that are less risk sensitive.

Viewing the CCP as a financial institution that holds a portfolio of clearing members’ portfolios, Section 2 considers the risk management problem that a CCP faces in the presence of clearing members’ variation and initial margin as a portfolio counterparty credit risk problem. Then, Section 2 uses the static copula threshold portfolio credit risk approach to approximate the CCP’s portfolio counterparty credit risk. Next, based on the one-period model, a two-step procedure is introduced. This procedure first specifies the total prefunded default funds of the CCP based on a particular risk measure and then uses the well known Euler principle to allocate the prefunded default fund to clearing members. After the prefunded default funds are specified, Section 2 uses the one-period model to define and formulate the unfunded default funds. The CCPs’ strict margin requirements are often criticized due to the destabilizing procyclicality they may cause during times of financial stress. Section 3 illustrates how margins’ procyclicality can be reduced in the one-period model. Section 4 derives the total unanticipated potential losses incurred by clearing members due to their CCP exposures and then illustrates how the CCP risk capital can be rigorously defined based on the unanticipated total losses. Section 5 discusses the current regulatory CCP risk capital framework and illustrates some of the challenges arising due to the absence of a model-based CCP risk measurement framework. Section 6 outlines how all the layers of the CCP default waterfall in the proposed one-period model can be estimated with Monte Carlo simulation in practice. The main purpose of Section 6 is to illustrate that some of the Monte Carlo procedures that have been successfully developed in the past decade can be used in our framework.

2 One-Period Models of Derivatives CCPs

Suppose that at a fixed time, \( T > 0 \), clearing member \( i \), \( CM_i \), defaults with probability \( p_i \); \( i = 1, ..., n \). The Bernoulli random variables \( Y_1, Y_2, ..., Y_n \) are the default indicators in the one-period model; \( E[Y_i] = P(Y_i = 1) = p_i, \ i = 1, ..., n \). Clearing member default indicators are made dependent by being defined as functions of \( n \) dependent underlying random variables, \( X_1, ..., X_n \), using copula threshold models, (see, e.g., Chapter 8 of McNeil et al. [2005]). Based on the foundational work of Merton [1974], \( CM_i \) defaults at time \( T \) if \( X_i \) crosses a given threshold value denoted by \( x_i \). In a \( t \)-copula threshold model specified below, \( X_1, ..., X_n \), are dependent Student \( t \) random variables,
\[ Y_i = 1 \{ X_i > x_i \} \text{, and } X_i = \frac{\sum_{j=1}^{d} a_{ij} Z_j + a_{i0} \xi_i}{\lambda} \; ; \; i = 1, ..., n, \]  

(1) 

where \( x_i \)'s are chosen to match the marginal default probabilities, \( p_i \). The correlation among \( X_i \)'s are specified through a \textit{factor model}, where the \textit{common risk factors}, \( Z_1, ..., Z_d \), and the \textit{idiosyncratic risk factor}, \( \xi_i \), are independent standard normal random variables, and \( a_{i0}, a_{i1}, ..., a_{id} \) are chosen such that the numerator becomes a standard normal random variable; \( \lambda \equiv \sqrt{\frac{K}{v}} \), and \( K \) has distribution \( \chi^2_v \) and is independent of \( Z_1, ..., Z_d \), and \( \xi_i \)'s. The \( t \)-copula model above becomes a normal copula by setting \( \lambda \) equal to 1 in which case \( X_1, ..., X_n \) become correlated normal random variables.\(^4\) Typical credit loss distributions are skewed with a relatively heavy upper tail; it is well-known that these empirical properties can not be captured by normal copula models as they – compared to other copula models, e.g., \( t \)-copulas – assign very small probabilities to simultaneous defaults, (see Sections 8.3.5 and 8.4.6 of McNeil et al. [2005] and the references there). As will be illustrated in the sequel, the CCP’s default waterfall layers beyond variation and initial margin are defined based on clearing members default indicators. Simultaneous defaults of clearing members and the tail of the CCP’s loss distribution should be modeled properly. Otherwise, CCP default waterfall resources could be underestimated; \( t \)-copula threshold models are preferable to normal copula threshold models in the proposed CCP risk measurement framework.\(^5\) 

Let \( C_i \) denote the CCP’s collateralized credit exposure to \( CM_i \) at its default in the presence of the \( CM_i \)'s variation and initial margins that have been posted to the CCP. The credit risk capital has been historically defined based on one-period models, where the exposure at default of an obligor in a loan portfolio is usually approximated by a constant. Basel II-III and risk capital researchers have also adapted one-period models for the counterparty credit risk capital by using the \textit{loan equivalent approach} and average-type dynamic counterparty risk measures such as expected positive exposure (EPE) or effective EPE as loan-equivalent exposures at default, (see Chapter 14 of Crouhy et al. [2001] on the loan equivalent approach, Pykhtin and Zhu [2006] for application of this approach in Basel II, and Gregory [2010] for various risk measures used in counterparty credit risk management).\(^6\) We use the similar approach and define \( C_i \)'s based on the loan-equivalent exposures at default for the proposed one-period model.

Specifically, let \( V_i^+(t) \equiv \max\{V_i(t), 0\} \) denote the positive part of the value of the derivatives portfolio that the CCP holds with \( CM_i \) at time \( t > 0 \), \( i = 1, ..., n \). The CCP’s collateralized exposure at time \( t \) in the presence of variation margin and initial margin can be defined as follows,

\[ e_i(t) \equiv \max\{V_i^+(t + \Delta) - VM_i(t) - IM_i(t), 0\}, \]  

(2) 

\(^4\)The one-factor normal copula threshold model was first introduced by Vasicek [1987] for loan portfolio modeling. 
\(^5\)\( t \)-copula threshold models can be calibrated similar to the widely used normal copula models as a multivariate \( t \) random vector is in fact a multivariate normal vector with a randomly scaled covariance matrix. 
\(^6\)Some authors use the term \textit{credit-equivalent} approach (exposures) for loan-equivalent approach (exposures).
where the VM available to the CCP at time $t$ conditional on $CM_i$’s time-$t$ default, depends on the frequency of variation margin calls denoted by $\Delta$. More specifically, $VM_i(t) \equiv V_i^+(t - \Delta)$.

Often, $\Delta$ is set equal to 1 day. The time interval, $\Delta + \Delta$, is usually pre-specified and is referred to as the margin period of risk (MPOR). MPOR is to represent the amount of time from the last VM call that a CCP requires to replace a defaulting clearing member’s derivatives portfolio, and it is usually set equal to 5 days for derivatives CCPs. In fact, given the concept of MPOR, it will be more conservative to define the uncollateralized replacement cost of the CCP due to a clearing member default at time $t$ as $\max_{t \leq u \leq t + \Delta} \{V_i^+(u)\}$. However, as $V$ could represent the value of hundreds or possibly thousands of derivatives contracts, accurate estimation of this maximum is impossible in practice. So, $V_i^+(t + \Delta)$ is used for the uncollateralized time-$t$ replacement cost.

Initial margin is a widely used risk mitigant for CCPs that has also been recently imposed on OTC derivatives market participants for non-centrally cleared transactions (see BCBS and IOSCO [2012]). IM is often defined based on a particular risk measure, e.g. value at risk (VaR) or expected shortfall (ES), associated with the random variable $R_i \equiv V_i^+(t + \Delta) - V_i^+(t - \Delta)$. Given some confidence level $\tilde{\alpha} \in (0, 1)$, the VaR-based and ES-based $IM_i \equiv IM_i(t)$ are,  

$$IM_i \equiv \text{VaR}_{\tilde{\alpha}}(R_i) = \inf \{r \in \mathbb{R} : P(R_i > r) \leq 1 - \tilde{\alpha}\}, \quad IM_i \equiv \text{ES}_{\tilde{\alpha}}(R_i) = E[R_i | R_i \geq \text{VaR}_{\tilde{\alpha}}(R_i)]$$

Then, the CCP’s EPE-based time-$T$ loan-equivalent collateralized exposure due to the $CM_i$’s default becomes,

$$C_i \equiv (1 - \delta_i)EPE_i = (1 - \delta_i) \int_0^T E[e_i(t)]dt, \quad (3)$$

where $\delta_i$ is the recovery rate of $CM_i$. For simplicity, we assume that $\delta_i$’s are constant. Similarly but more conservatively, $C_i$, can be defined based on effective EPE ,

$$\text{eEPE}_i = \int_0^T \max_{0 \leq u \leq t} E[e_i(u)]dt.$$  

Monte Carlo simulation is widely used to estimate various counterparty credit risk measures, (see, e.g., Ghamami and Zhang [2013]). Section 6 summarizes similar ideas for Monte Carlo EPE or eEPE-based estimation of $C_i$’s. Loan-equivalent collateralized exposures need not be defined based on average-type counterparty credit risk measure. For instance,

$$\int_0^T \text{VaR}_{\tilde{\alpha}}(e_i(t))dt,$$

can be considered as a quantile-based dynamic measure of counterparty credit risk at a given confidence level, $\tilde{\alpha}$, and can be used to define $C_i$’s.  

\footnote{For a discontinuous loss distribution function, the widely used definition of ES does not hold for all $\tilde{\alpha}$. The following term should be added to the ES in the continuous case, $(1 - \tilde{\alpha})^{-1} \text{VaR}_{\tilde{\alpha}}(1 - \tilde{\alpha} - P(R_i \geq \text{VaR}_{\tilde{\alpha}})).$}

\footnote{CCPs can auction the defaulting member’s portfolio to other clearing members. This feature does not exist for
Note that, for simplicity, it has been assumed that the credit quality of the clearing members does not depend on the value of the cleared derivatives portfolios. That is, the so called dependent risk – wrong way or right way risk – is not captured in the above formulation, (see Ghamami and Goldberg [2014] for subtleties involved in modeling dependent risk in practice). This can be seen by referring to (1) and (3), and noting that the CCP’s derivatives portfolio values with $CM_i$’s, i.e., $V_i$’s, are defined independent from $CM_i$’s credit qualities represented by $p_i$’s.

Section 2.1 illustrates how prefunded default fund contributions, $DF_1, ..., DF_n$ are specified based on the CCP’s loan-equivalent collateralized exposure to clearing members and default indicators, $(C_1,Y_1), ..., (C_n,Y_n)$, under the proposed one-period model. For the remainder of this section suppose that prefunded default fund contributions are given. Let $U_i$ denote the CCP’s credit exposure to $CM_i$ at its default that has been further mitigated by the $CM_i$’s prefunded default fund contribution. That is,\footnote{\textit{As will be shown in the next section, $DF_i$ is defined based on $(C_1,Y_1), ..., (C_n,Y_n)$ such that $C_i \geq DF_i$ for all $i = 1, ..., n$. So, the one period model gives $U_i = (C_i - DF_i)^+ = C_i - DF_i$. In this section, however, one can think of $U_i$’s as model independent quantities.}}

\[
U_i = (C_i - DF_i)^+.
\]

Recall the CCP’s default waterfall, the aggregate potential loss to the CCP after the defaulter-pay resources, $VM, IM$ and $DF$, are used, is

\[
L^{(1)} = \sum_{i=1}^{n} U_i Y_i.
\]

This loss is then to be protected by the CCP’s equity contribution, $E$, and the survivor-pay prefunded default fund contributions. The aggregate second-level potential loss, after the survivor-pay resources and the CCP’s equity contribution are used, becomes,

\[
L^{(2)} = \left( \sum_{i=1}^{n} U_i Y_i - E - DF_s \right)^+,
\]

where,

\[
DF_s \equiv DF - \sum_{i=1}^{n} DF_i Y_i,
\]

is the prefunded default fund contribution of the surviving members and $DF \equiv DF_1 + ... + DF_n$. Note that $L^{(2)}$ represents the loss that would exceed both the defaulter-pay resources and the prefunded default funds of the surviving members. $L^{(2)}$ or part of it is allocated to the surviving members in the form of the CCP’s unfunded default fund capital calls.

Next section introduces a 2-step procedure that specifies the prefunded default funds in the one-period model. Using the model-based characterization of $DF_1, ..., DF_n$, Section 2.2 shows how the unfunded default funds are specified.
2.1 A Two–Step Procedure to Define the Prefunded Default Funds

The regulatory CCP risk management framework is developed based on a set of broad principles referred to as the *Principles for Financial Market Infrastructures* (PFMI), (see CPSS and IOSCO [2012]). Derivatives CCPs often specify their total prefunded default funds, $DF$, based on the so called *Cover 1/Cover 2* principle of PFMI, which is summarized below:\(^{10}\)

*A systemically important CCP or a CCP that is involved in activities with complex risk profiles should maintain financial resources to cover the default of two participants that would potentially cause the largest aggregate credit exposure for the CCP in extreme but plausible market conditions. Other CCPs should maintain financial resources to cover the default of the participant that would potentially cause the largest aggregate credit exposure for the CCP in extreme but plausible market conditions.*

Unlike the variation and initial margin, the PFMI’s treatment of the prefunded default funds is broad and qualitative, and so a unifying and model-based specification of the prefunded default funds across derivatives CCPs becomes very difficult. Even if before the clearing mandate, CCPs could have some discretion in specifying their default waterfall resources beyond variation and initial margin, the regulatory CCP risk capital requirements on clearing members may create an ambiguous central clearing environment if prefunded and unfunded default funds continue to be qualitatively specified, (see and compare BCBS [2012] and CPSS and IOSCO [2012]). As will be shown in Sections 4 and 5, clearing members’ CCP risk capital depends on prefunded default funds; inconsistencies in how these funds are specified and allocated to clearing members will directly affect the CCP risk capital requirement of clearing members. As will be shown in this paper, at least three ambiguous aspects of the *Cover 1/Cover 2* principle can be dealt with by introducing a mathematical model for the default waterfall resources of derivatives CCPs. First, when $n$, the number of clearing members, becomes relatively large and the portfolios that the clearing members hold with the CCP are relatively homogeneous, it is not clear whether defining the $DF$ based on the *Cover 1/Cover 2* principle would protect the CCP sufficiently well compared to the case where the CCP has a smaller number of clearing members with more heterogenous portfolios. Secondly, the credit quality of the clearing members and the correlation among them do not play any role in the *Cover 1/Cover 2* principle. Thirdly, the allocation of $DF$ to clearing members has remained a subjective matter among derivatives CCPs.

Introducing ideas from the financial risk management literature, this paper proposes a two-step procedure for specifying $DF$ and $DF_1, ..., DF_n$ under the one-period model. First, define the total prefunded default fund, $DF$, based on a particular risk measure – value at risk (VaR) or expected shortfall (ES) – associated with the total loss in the presence of only variation and initial margins. The CCP’s total credit loss is denoted by $L$ and specified as follows,

\(^{10}\)CCP’s are considered to be a particular type of FMI’s by the PFMI.
\[ L = \sum_{i=1}^{n} C_i Y_i, \]  

(5)

where \( C_i \), defined in (3), is the CCP’s loan-equivalent exposure to \( CM_i \) at its default under the one-period model. Given some confidence level \( \alpha \in (0, 1) \), the VaR-based \( DF \) becomes,

\[ DF \equiv VaR_\alpha(L) \equiv \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\}, \]  

(6)

and the ES-based \( DF \) for the confidence level \( \alpha \) becomes,

\[ DF \equiv ES_\alpha(L) \equiv ES_\alpha = E[L|L \geq VaR_\alpha(L)]. \]  

(7)

It is well known from the axiomatic characterization of risk measures that ES is preferable to VaR as the former is a coherent risk measure while the latter is not, (see Chapter 6 and proposition 6.9 of McNeil et al. [2005]).

In the one-period model of central counterparty risk, the total prefunded default funds, \( DF \), of a CCP is defined similar to the way the portfolio credit risk capital is defined for a loan portfolio. The CCP can be in fact viewed as a financial institution exposed to its portfolio counterparty credit risk, where the portfolio constituents are the CCP’s clearing members’ derivatives portfolios, and the CCP’s exposure to each clearing member is collateralized by both variation and initial margin. The static model is then a one-period approximation of the portfolio counterparty credit risk as the potential default of each clearing member is allowed to occur at a single given time point denoted by \( T \), (both Basel II and III use this approximation in defining the counterparty credit capital at a single counterparty level).

As it can be seen from the right side of (6) or (7), the prefunded default fund, \( DF \), which is defined based on a risk measure associated with portfolio counterparty credit risk, depends on the collateralized exposures of the CCP to each clearing member and the credit quality of clearing members as represented by default indicators.

The Euler capital allocation principle, which has been successfully used in finance (see Chapter 6 of McNeil et al. [2005] et al. and the reference there), can be employed in the second step to allocate \( DF \) to clearing members and to specify their prefunded default fund contributions \( DF_1, ..., DF_n \). More specifically, for the VaR-based \( DF \), the prefunded default fund contribution of \( CM_i \) based on the Euler allocation principle becomes,

\[ DF_i = C_i E[Y_i|L = VaR_\alpha], \]  

(8)

and when \( DF \) is defined based on \( ES \), the Euler allocation principle specifies \( DF_i \) as follows,

\[ DF_i = C_i E[Y_i|L \geq VaR_\alpha]. \]  

(9)

Euler allocation rules are defined based on the partial derivatives (sensitivities) of the risk measure under consideration and their derivation depends on technical conditions specified in
Tasche [1999], (see, also, Section 6.3 of McNeil et al. [2005]). Note that both allocation rules satisfy the so called full allocation property. That is, in the case of VaR-based $DF$, we can write,

$$\sum_{i=1}^{n} DF_i = \sum_{i=1}^{n} C_i E[Y_i | L = VaR_{\alpha}] = E[L | L = VaR_{\alpha}] = VaR_{\alpha} = DF,$$

and for ES-based DF, we have,

$$\sum_{i=1}^{n} DF_i = \sum_{i=1}^{n} C_i E[Y_i | L \geq VaR_{\alpha}] = E[L | L \geq VaR_{\alpha}] = ES_{\alpha} = DF.$$

The full allocation property, a desirable property of any allocation principle, simply means that the total prefunded default fund is fully allocated to clearing members.

This paper proposes to form the total prefunded default fund based on the expected shortfall as in (7) and then allocate it using the Euler principle based on (9). Section 6 discusses how Monte Carlo simulation can be used to estimate $DF, DF_1, ..., DF_n$.

When the total prefunded default fund is specified and allocated under our proposed two-step procedure, the CCP’s second-level loss accepts the following useful expression,

$$L^{(2)} = \left( \sum_{i=1}^{n} C_i Y_i - E - DF \right)^+. \tag{10}$$

The derivation of the right side above uses the model independent definition of $L^{(2)}$ given in (4) and that based on the proposed prefunded default fund allocation rules in (8) and (9), we have $DF_i = C_i E[Y_i | O] \leq C_i$, with $O$ denoting the event $\{L \geq VaR_{\alpha}\}$ or $\{L = VaR_{\alpha}\}$. Using the confidence level $\alpha$ associated with the VaR or ES-based $DF \equiv DF_{\alpha}$ and the CCP’s equity contribution $E$, the probability of the event $\{L > E + DF\}$ can be made very small. That is, since $P(L > E + DF) \leq 1 - \alpha$, depending on $E$ and $DF_{\alpha}$, the CCP’s potential loss that finds its way to the last layer of the default waterfall resources, i.e., the unfunded default funds, can be made very small.

**Remark 1** Capital allocation is a well-studied topic in finance; we will not present an exhaustive literature survey on it in this paper. We refer only to a few influential papers on economic justification of the Euler allocation principle and on the comparison between VaR and ES in portfolio credit risk; all are applicable in our setting as we have taken a portfolio credit risk approach to define and allocate $DF$. Tasche [1999] gives the first economic justification of the Euler principle based on a return on risk-adjusted capital (RORAC) criterion, which is also applicable to portfolio credit risk. Denault [2001] uses cooperative game theory for an economic justification of the Euler principle. The axiomatic characterization of risk measures has been well received in the finance community; Kalkbrener [2005] introduces an axiomatic approach that formalizes a set of properties for good allocation rules; then, under some technical conditions,
the Euler principle proves to be the only allocation rule having the formalized desirable properties. Kalkbrener et al. [2004] and Kurth and Tasche [2003] argue that ES is preferable to VaR particularly for portfolio credit risk; VaR does not encourage diversification for portfolio credit risk. Also, a set of numerical examples in Kalkbrener et al. [2004] and Kurth and Tasche [2003] indicate that ES contributions (see the right side of (9)) are more sensitive to concentration risk compared to VaR contributions (see the right side of (8)).

Remark 2 Suppose that the CCP stands between two counterparties to a derivatives transaction, e.g., an interest rate swap. If both clearing members default at the same time, the CCP, by its definition, would not incur any direct losses. In the proposed one-period model, however, when the above-mentioned matched pair of clearing members default, the loss to the CCP in the presence of only VM and IM would be the sum of $C_1 \geq 0$ and $C_2 \geq 0$, where $C_1$ and $C_2$ denote the CCP’s loan-equivalent exposure to these clearing members at their default. That is, the one-period model does not capture this characteristic of a CCP’s theoretical arrangement. In this sense the model produces conservative estimates for the default waterfall resources beyond variation and initial margin.

2.2 The Unfunded Default Funds

Consider two cases. When $L^{(2)}$, the CCP’s potential loss exceeding variation and initial margins, CCP’s equity contribution, and the prefunded default funds,

$$L^{(2)} = \left( \sum_{i=1}^{n} C_i Y_i - E - DF \right)^+, \tag{11}$$

is fully allocated to the surviving members, the CCP’s unfunded default fund capital calls on clearing members are referred to as uncapped. Otherwise, when the clearing members’ unfunded default funds are capped by a multiple of their prefunded default funds, the CCP’s unfunded default fund capital calls are capped. In the former case, we assume that $L^{(2)}$ is allocated to the surviving members proportional to their prefunded default fund contributions.\footnote{This is a common assumption in practice. We discuss the risk management justification of this allocation rule in Appendix A.} That is, the CCP’s uncapped unfunded default fund capital call on clearing member $i$ becomes,

$$\tilde{DF}^{uc}_i \equiv L^{uc}_i = \frac{DF_i (1 - Y_i)}{DF_s} L^{(2)} \geq 0,$$

where $DF_s = \sum_{j=1}^{n} DF_j (1 - Y_j)$. Note that distributing $L^{(2)}$ proportional to prefunded default fund contributions gives,

$$\tilde{DF}^{uc} \equiv \tilde{DF}^{uc}_1 + \ldots + \tilde{DF}^{uc}_n = L^{(2)}.$$

$L^{(2)}$ can also be allocated to the surviving members proportional to the loan-equivalent exposures, i.e., $C_i$’s. However, allocation of $L^{(2)}$ proportional to the $C_i$’s will not capture the credit...
quality, correlation, and tail dependence of the clearing members in a sense that is discussed in Appendix A. While the unfunded default funds of surviving clearing members are the CCPs’ last layer of the default waterfall resources, they are unanticipated potential losses to surviving members. The double notation $\tilde{DF}_{i}^{uc} \equiv L_{i}^{uc}$ is to emphasize this point and simplify the communication of our results when moving from the CCP’s perspective to that of a clearing member. When the $CM_{i}$’s unfunded default fund is capped by a multiple of its prefunded default fund contribution, it becomes,

$$\tilde{DF}_{i} \equiv L_{i} = \min\{L_{i}^{uc}, \beta DF_{i}(1 - Y_{i})\},$$

(12)

where $\beta > 0$.

Clearly, the uncapped and capped unfunded default funds of $CM_{i}$ are zero conditional on its default at time $T$. As will be shown in the next section, the CCP risk capital of clearing members depends on the unfunded default funds; the $CM_{i}$’s CCP risk capital is an increasing function of $L_{i}^{uc}$. From the bank regulators’ and the $CM_{i}$’s perspective, it will be more conservative to specify the $CM_{i}$’s CCP risk capital by defining the unfunded default funds assuming that $CM_{i}$ survives at time $T$. That is, the $CM_{i}$’s uncapped unfunded default fund assuming its time-$T$ survival is defined as follows,

$$\tilde{DF}_{i}^{uc,s} \equiv L_{i}^{uc,s} = \frac{DF_{i}}{DF_{s,i}} \left( \sum_{j \neq i} U_{j}Y_{j} - E - DF_{s,i} \right) + \frac{DF_{i}}{DF_{s,i}} \left( \sum_{j \neq i} C_{j}Y_{j} - E - DF_{j} \right),$$

(13)

where $DF_{s,i}$ denotes the surviving members’ prefunded default funds under the assumption that $CM_{i}$ survives at time $T$,

$$DF_{s,i} \equiv DF - \sum_{j \neq i} DF_{j}Y_{j},$$

(14)

and to derive the right side of (13), we use simple algebraic manipulations used in the derivation of (10). Similarly, assuming the $CM_{i}$’s time-$T$ survival, the CCP’s unfunded default fund capital call on $CM_{i}$ becomes,

$$\tilde{DF}_{i}^{s} \equiv L_{i}^{s} = \min\{L_{i}^{uc,s}, \beta DF_{i}\},$$

(15)

where $\beta > 0$.

\textsuperscript{12}To simplify the notation we have written $\tilde{DF}_{i}$ and $L_{i}$ for $\tilde{DF}_{i}^{c}$ and $L_{i}^{c}$, respectively – superscript $c$ denoting the capped case.
Remark 3  After the prefunded default funds are specified by the CCP, in defining $L_{i}^{uc,s}$ under the assumption that $CM_i$ survives at time $T$, one should think of $CM_i$ as the financial institution that, independent of its own credit quality, considers the credit quality of $CM_j$’s, $j \neq i$, and represents them by dependent default indicators $Y_j$, $j \neq i$. Note that the two random variables $L_{i}^{uc,s}$ and $L_{i}^{uc}$ conditional on $Y_i = 0$ are not equal in distribution as $Y_i$’s are dependent random variables. That is, for any $x > 0$,

$$P(L_{i}^{uc} \leq x | Y_i = 0) \neq P(L_{i}^{uc,s} \leq x).$$

Remark 4  Given the proposed two-step procedure to specify $DF$ and decompose it to $DF_i$’s based on VaR or ES, we have,

$$\tilde{DF}_i^s \equiv L_i^s = \min \left\{ \frac{DF_i}{DF_{s,i}} \left( \sum_{j \neq i} C_j Y_j - E - DF \right)^+, \beta DF_i \right\},$$

Note that

$$P \left( \sum_{j \neq i} C_j Y_j > E + DF \right) \leq P(L > DF) \leq 1 - \alpha,$$

where $\alpha$ denotes the confidence level associated with $DF$. So, depending on $DF_\alpha$ and $E$, the probability that a surviving member would incur a loss at time $T$ can be made very small.

3 The One-Period Model and Margins’ Procyclicality

The CCPs’ variation and initial margin requirements are often criticized as they might lead to destabilizing feedback mechanisms, (see, e.g., Pirrong [2013]). During times of financial stress, the market volatility increases. This in turn increases the initial margin requirements. High margin requirements during financial stress affect the funding and market liquidity, (see Brunnermeier and Pedersen [2009]); liquidity dry-ups further increase the market volatility. The resulting feedback mechanism can exacerbate and prolong the down times. There has been several regulatory proposals to mitigate the procyclicality associated with margin requirements. For instance, financial institutions subject to regulatory risk capital are being asked to use longer historical data containing stressed periods in the calibration of their margin models, (see, e.g., CGFS [2010]). Also, Basel III has devised capital conservation and counter-cyclical buffers in its capital requirements, (see, e.g., Chapter 13 of Hull [2012]).

The proposed default waterfall model can be used to reduce margins’ procyclicality in a more risk sensitive way. To show this, we first summarize the default waterfall layers in the one-period model. Recall a CCP’s time-$T$ collateralized exposure due to the $CM_i$’s default,$^{14}$

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$^{13}$CGFS stands for the Committee on the Global Financial System.

$^{14}$As stated in Section 2, $C_i$’s can also be defined based on other dynamic counterparty credit risk measures.
\[ C_i = (1 - \delta_i) \int_0^T E[e_i(t)]dt, \]
as defined in (3) with \( e_i(t) \equiv \max\{V_i^+(t+\Delta) - VM_i(t) - IM_i(t), 0\}; \quad i = 1, \ldots, n; \) the IM is based on a given confidence level, \( \tilde{\alpha} \). The total prefunded default fund is specified and allocated to clearing members as follows,

\[ DF_\alpha \equiv E[L|L > q], \quad \text{and} \quad DF_i \equiv C_i E[Y_i|L > q], \]

with \( L = \sum_{i=1}^n C_iY_i \) and \( q \equiv VaR_\alpha(\sum_{i=1}^n C_iY_i) \). The CCP’s potential loss in the presence of variation margin, initial margin, CCP’s equity contribution, and prefunded default funds is,

\[ L^{(2)} = \left( \sum_{i=1}^n C_iY_i - E - DF_\alpha \right)^+. \]

Recall the CCP’s unfunded default fund capital call on \( CM_i \) in the capped case,

\[ \tilde{DF}_i = \min \left\{ \frac{DF_i(1-Y_i)}{DF_s}L^{(2)}, \beta DF_i(1-Y_i) \right\}, \]

as defined in (12) and (11) with \( DF_s = \sum_{j=1}^n DF_j(1-Y_j) \). Let \( \tilde{DF} \equiv \sum_{i=1}^n \tilde{DF}_i \). Consider the CCP’s expected potential loss in the presence of all layers of the default waterfall resources,

\[ E[L^{(3)}] = E[(\sum_{i=1}^n C_iY_i - E - DF_\alpha - \tilde{DF})^+]. \]

Given the one-period model’s characterization of the default waterfall resources, the CCPs and their regulators can reduce margins’ procyclicality by lowering margin requirements while preserving the same expected loss, \( E[L^{(3)}] \). Margin requirements can be reduced by, for instance, lowering the frequency of variation margin calls and the confidence level associated with members’ initial margin. That is, the mix of margin requirements, CCPs equity contribution, and prefunded and unfunded funds can be chosen such that margins’ procyclicality be reduced while the CCPs maintain the same level of financial resiliency. As will be shown in the next section, the CCPs’ unfunded default fund capital calls are unanticipated losses on surviving members, against which bank regulators require risk capital on clearing members. Clearly, if the default waterfall is shifted heavily toward the unfunded default funds, these unanticipated losses during times of financial stress can become new sources of procyclicality. Also, high CCP risk capital charges on clearing members can disincentivize the central clearing of OTC derivatives transactions. To mitigate the margins’ destabilizing feedback mechanism, CCPs can lower their margin requirements by increasing the prefunded default funds and the CCP’s equity contribution to maintain the same level of expected second level loss, \( E[L^{(2)}] \), and so the same level of expected unanticipated losses to clearing members.
4 The CCP Risk Capital of Clearing Members

Credit risk is the classical risk borne by a bank that is in the business of deposit taking and loan making. The Basel Committee on Banking Supervision (BCBS) has defined and required regulatory credit risk capital for more than two decades. Regulatory counterparty credit risk capital has also been devised for protection against potential counterparty credit losses.\(^\text{15}\) From the bank regulators’ perspective, direct clearing members of CCPs should hold CCP risk capital against their central counterparty credit exposures. The CCP risk capital of clearing members are based on two types of losses: the unanticipated losses due to the default of other clearing members that exceed clearing members’ VM, IM, the CCP’s equity contribution, E, and DF, i.e., the CCPs’ unfunded default fund capital calls and the potential loss due to the default of the central counterparty itself. Upon the CCP’s default, each clearing member may incur a replacement cost based on the value of the derivatives portfolio it holds with the CCP.

Appendix B formulates the CCP’s default probability in the one-period model. The one-period model assumes that the CCP defaults when an uncovered loss remains at time \(T\) after all the CCP’s default waterfall resources are exhausted. Using the results of Appendix B, Section 4.1 derives a clearing member’s total potential loss due to its CCP exposures. Section 4.2 defines the CCP risk capital using expected and unexpected losses under the static model.

4.1 Total Losses to the Clearing Members

Hereafter, in the main body of the paper, we assume that a clearing member’s total potential loss is defined under the assumption that it survives at time \(T\). Appendix C derives the clearing member total losses in the absence of this assumption.

In the uncapped case, the CCP may default only when all clearing members default at time \(T\). So, assuming \(CM_i\)’s survival at time \(T\), the total potential loss in the uncapped case is equal to the \(CM_i\)’s unfunded default fund,

\[
L_{i,uc,s}^{uc} = L_{i,uc}^{uc} = \frac{DF_i}{DF_{s,i}} \left( \sum_{j \neq i} C_j Y_j - E - DF \right)^+, \tag{16}
\]

where \(DF_{s,i} \equiv DF - \sum_{j \neq i} DF_j Y_j\) as defined in (13)–(14). When the \(CM_i\)’s unfunded default fund is capped by a multiple of its prefunded default fund, the total potential loss is due to the possible default of other clearing members and the CCP. Let \(\tilde{Y}_i\) denote the CCP’s default indicator from \(CM_i\)’s perspective assuming it’s survival at time \(T\). In the capped case, the \(CM_i\)’s total potential loss becomes,

\[
L_{i,s} = L_s^i + \tilde{U}_i \tilde{Y}_i, \tag{17}
\]

\(^{15}\)A counterparty in the above mentioned context refers to an OTC derivatives market participant that holds a derivatives portfolio with the financial institution under regulatory requirements.
where $L_s = \min \{ L_s^{uc,s}, \beta DF_i \}$, which is defined in (15), denotes the $CM_i$’s potential loss due to the possible default of other clearing members. The CCP’s default probability assuming $CM_i$’s time-$T$ survival, $P_{ccp,i} = E[\tilde{Y}_i]$, is derived in (26) of Appendix B. Also, $\tilde{U}_i$ denotes the $CM_i$’s loan-equivalent exposure to the CCP at its default. Recall (3); an EPE-based exposure at default is defined as,

$$\tilde{U}_i = (1 - \delta) \int_0^{\tilde{T}} E[\tilde{e}_i(t)]dt,$$

where $\delta$ denotes the CCP’s recovery rate, $\tilde{e}_i(t)$ is $CM_i$’s collateralized exposure to the CCP at time $t$ in the presence of variation margin, and $\tilde{T}$ specifies the time interval based on which clearing members’ loan-equivalent exposure is estimated. CCPs do not usually post initial margin to the clearing members. More specifically,

$$\tilde{e}_i(t) \equiv \max \{ \tilde{V}_i^+(t) - \tilde{VM}_i(t), 0 \},$$

where $\tilde{V}_i^+(t)$ is nonnegative part of the value of the derivatives portfolio that $CM_i$ holds with the CCP at time $t$, and the VM available to the $CM_i$ at time $t$ depends on the frequency based on which the CCP posts variation margin to the clearing members. Let $\tilde{\Delta}$ denote the length of the time interval associated with the VM calls, then, $VM_i(t) \equiv V_i^+(t - \tilde{\Delta})$.

When the CCP’s unfunded default fund capital calls are capped, the loss decomposition formula in (17) illustrates that the total loss of a clearing member is the sum of the loss caused by other members’ default and the loss due to the CCP’s default.

### 4.2 The CCP Risk Capital

The definition of the portfolio credit risk capital is often based on unexpected losses. When the one-period model is used for modeling the total loss, $\tilde{L}$, associated with a loan portfolio, the VaR-based unexpected portfolio credit risk capital at a given confidence level, $\alpha$, becomes $VaR_\alpha(\tilde{L}) - E[\tilde{L}]$, (see, e.g., Chapter 12 of Hull [2012]). Differentiating expected and unexpected losses in the context of counterparty credit risk (CCR) is not straightforward. Basel II’s counterparty risk capital is based on EPE or eEPE times the counterparty’s default probability defined using a one-period model, (see, e.g., Pykhtin and Zhu [2006]). Basel III’s CCR capital requirements consist of credit value adjustment (CVA) and Basel II’s CCR risk capital. CVA, in contrast to Basel II’s CCR risk capital, is not estimated based on one-period models; the definition of CVA allows the counterparty to default at any point in time in a given time interval. So, it is difficult to define a rigorous framework for Basel III’s counterparty credit risk capital differentiating expected and unexpected losses (see, e.g., Pykhtin [2011]).

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16 The time integral of expected exposures are also used to define the counterparty risk capital in Basel II and III. Since the credit quality of CCPs are usually viewed higher than other derivatives market participants, clearing members’ loan-equivalent exposure to CCPs are based on shorter time intervals when compared to the non-central clearing loan-equivalent exposures.

17 In cases where upon the CCP’s default, the surviving members’ initial margins are not returnable, (4.1) can be modified as follows, $\tilde{e}_i(t) \equiv \max \{ \tilde{V}_i^+(t) - VM_i(t), IM_i(t), 0 \}$, where $IM_i(t)$ is defined in Section 2.
Using the proposed one-period model, the CCP risk capital of direct clearing members can be defined based on their expected or unexpected total potential losses. Consider the CCP risk capital from a clearing member’s perspective assuming its time-$T$ survival and recall the $CM_i$’s total loss decomposition in the capped case, $L_{t,s}^{i,s} = L_{t}^{s} + \tilde{U}_{i} Y_{i}$. The CCP risk capital of $CM_i$ based on total expected capped losses becomes,

$$E[L_{t,s}^{i,s}] = E[L_{t}^{s}] + \tilde{U}_{i} P_{ccp,i},$$

where the CCP’s default probability from the $CM_i$’s standpoint, $P_{ccp,i}$, is derived in (26) of Appendix B, and $\tilde{U}_{i}$, the $CM_i$’s loan-equivalent exposure to the CCP at its default, is given in (18). Or, the $CM_i$’s CCP risk capital can be based on unexpected losses,$^{18}$

$$VaR_{\alpha}(L_{t,s}^{i,s}) - E[L_{t,s}^{i,s}],$$

given a confidence level $\alpha \in (0,1)$. Section 6 outlines Monte Carlo estimation of CCP risk capital. Appendix D approximates clearing members expected uncapped losses for derivatives CCPs with a large number of clearing members. The approximation is based on the widely used single-factor normal copula threshold model of portfolio credit risk.$^{19}$

5 The Current Regulatory CCP Risk Capital

Recognizing that banks differ in their level of sophistication to use mathematical models for risk capital calculations, the BCBS has often developed both non-model-based and more risk sensitive model-based methods for risk capital requirements. The credit risk framework of Basel II is a good example where the credit risk capital can be calculated based on the non-model-based Standardized approach, or based on the model-based more risk sensitive Foundation Internal Ratings Based (IRB) and the Advanced IRB approaches, (see, e.g., Chapter 12 of Hull [2012]).

The regulatory risk capital interim framework for banks exposures to CCPs was published by BCBS [2012]. A consultative document, BCBS [2013], has modified the interim framework and is supposed to be the basis for the finalized CCP risk capital.$^{20}$ None of the proposed CCP risk capital methods are based on a mathematical model of the CCPs default waterfall resources. That is, a model-based approach to CCP risk capital does not exist in the current regulatory framework. As it has been shown in this paper, clearing members’ CCP risk capital depends on all the layers of CCPs’ default waterfall resources in a complicated way. So, even a non-model-based less risk sensitive definition of the CCP risk capital in the absence of a rigorous CCP risk measurement framework is not straightforward. This section does not discuss the current

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$^{18}$The CCP risk capital can be also defined in a less conservative way based on unconditional total losses to clearing members derived in Appendix C. The term “unconditional” refers to the case where we do not assume the clearing member’s survival at time $T$.

$^{19}$Basel II’s internal rating based (IRB) approach for credit risk capital uses normal copula-based asymptotic results for large loan portfolios developed by Gordy [2003].

$^{20}$As stated before, BCBS is not the only international standard setting body responsible for CCP risk capital.
regulatory CCP risk framework in detail; it only highlights some of the difficulties in defining the CCP risk capital in the absence of a unifying CCP risk measurement framework.\textsuperscript{21}

As illustrated in Section 2.1, the PFMI's Cover 1/Cover 2 principle can not be used for a model-based specification of the total prefunded default fund and its allocation to clearing members. The ambiguity surrounding $DF$ and $DF_i$'s will carry over to CCP risk capital rules as they depend on the prefunded default funds, (see the loss decomposition formula (17)).

Before discussing the current regulatory framework, we first argue that a risk sensitive CCP risk capital method should differentiate the prefunded and unfunded default funds and should be based on the CCPs' unfunded default fund capital calls as they represent the clearing members' unanticipated losses due to the CCP exposures. The clearing members' variation and initial margins are updated on a frequent basis due to the changes in the members' derivative portfolio values over time. The VM and IM are clearing members' collateral posted to the CCP. Clearing members do not usually hold regulatory capital against the CCPs' margin requirements. Similarly, prefunded default funds are to be updated on a pre-specified basis due to the changes in the value of the derivatives portfolios, the credit quality of the clearing members, and also possible clearing member defaults. Similar to VM and IM, the prefunded default funds of a clearing member can be viewed as the member's second layer of collateral posted to the CCP. If some of the clearing members default over a given period of time, say over $[0,T]$ such that the resulting total loss would not exceed the total $DF$, the surviving member under consideration can respond to the CCP and update its prefunded default funds at or after time $T$, or it could cancel it's CCP membership in which case the CCP would not incur any additional direct loss. In contrast to the prefunded default funds, if a surviving member does not meet the CCP's unfunded default fund capital calls, this may lead to additional losses and ultimately the CCP's default. Therefore, the clearing members' CCP risk capital is to ensure that the CCPs' unfunded default fund capital calls are met. That is, the clearing members' CCP risk capital requirements exist primarily because of the CCPs' unfunded default fund capital calls and the CCPs' default risk.

The BCBS's CCP risk capital of direct clearing members consists of default fund capital charges due to the possible defaults of other clearing members and trade exposure capital charges due to CCP's default risk. As it is evident from the definition of default fund capital charges, the current regulatory framework does not differentiate prefunded defaults funds from unfunded default funds. In fact, it is not possible to differentiate the two in the absence of a default waterfall model. The BCBS [2013] proposes two methods to calculate default fund capital charges, where one method is supposed to be more risk sensitive than the other. (BCBS [2012] uses a similar approach). First, consider the more risk sensitive Tranches approach of BCBS [2013] for the default fund capital charge of clearing member $i$ denoted by $K_{cm_i}$.

\textsuperscript{21}This section does not intend to promote model-based risk sensitive approaches to the CCP risk capital over non-model-based methods. We are to show that a model-driven risk sensitive framework, which does not currently exist, can be used to better formulate and evaluate the less risk sensitive non-model-based methods.
\[ K_{cm_i} = \begin{cases} 
K_{ccp} - E & \text{if } DF + E < K_{ccp} \\
(K_{ccp} - E) + c_1(DF + E - K_{ccp}) & \text{if } E < K_{ccp} < E + DF \\
c_1 DF & \text{if } K_{ccp} < E,
\end{cases} \]

where \( c_1 = 0.16 \left( \frac{K_{ccp}}{DF + E} \right) \), and \( K_{ccp} \),

\[ K_{ccp} = (\text{capital ratio}) \times RW \times \left( \sum_{i=1}^{n} EAD_i \right), \]

is to represent the CCP’s *hypothetical capital requirements* due to its members’ counterparty credit risk exposures; \( EAD_i \) is the CCP’s exposure to \( CM_i \) at its default in the presence of variation and initial margin. The risk weight \( RW \) is usually set to .2, and the capital ratio has been set to .08. (The BCBS’s minimum capital ratio has been historically set to .08).\(^{22}\)

The BCBS has historically used *risk weights* for its less risk sensitive non-model based risk capital methods, (see, e.g., Chapter 2 of Crouhy et al. [2001]). Note that the non-model-based CCP’s *hypothetical capital*, \( K_{ccp} \), is a component of the Tranches approach, display (20), which is supposed to be the more risk sensitive method for the default fund risk capital calculations.

To simplify (20) and facilitate a comparison between the Tranches approach and CCP risk capital in the proposed framework, suppose that the CCP’s equity contribution, \( E \), is zero. Given the proposed one period model, set \( EAD_i \equiv C_i, i = 1, ..., n \). The Tranches approach then gives,

\[ K_{cm_i} = \begin{cases} 
K_{ccp} & \text{if } DF < K_{ccp} \\
K_{ccp} + \frac{16K_{ccp}(DF - K_{ccp})}{DF} & \text{if } K_{ccp} < DF,
\end{cases} \]

where \( K_{ccp} = 0.016 \times (\sum_{i=1}^{n} C_i) \). Note that the \( CM_i \)’s CCP risk capital increases when \( DF > K_{ccp} \). This can not be rationalized. In the less risk sensitive *Ratio* approach of BCBS [2013], the default fund capital charge of clearing member \( i \), denoted by \( K_{cm_i} \), is defined as follows,

\[ K_{cm_i} = \frac{DF_i}{DF} K_{ccp}, \quad (21) \]

\(^{22}\)The Tranches approach of BCBS [2013] uses \( RLDF = \max\{DF_{cover}^*, K_{ccp}\} \) for \( K_{ccp} \), where \( DF_{cover}^* \) denotes a CCP’s calculation of prefunded default fund contributions based on the *Cover 1/Cover 2* principle. For simplicity, \( DF_{cover}^* \) is not considered in Section 5; it merely adds to the conceptual ambiguities that Section 5 attempts to highlight.
where the risk weight in $K_{ccp} = 0.08 \times RW \times (\sum_{i=1}^{n} EAD_i)$ in the Ratio approach is larger than the risk weight in the Tranches approach.\textsuperscript{23}

The Tranches and Ratio approaches are risk insensitive as they are not based on the CCP’s unfunded default fund capital calls; both approaches mainly allocate the CCP’s total collateralized exposures, $C_1 + ... + C_n$, to clearing members proportional to their prefunded default funds, $DF_1, ..., DF_n$. The PFMI’s treatment of $DF_i$’s is broad and qualitative. This may lead to inconsistencies in CCP risk capital calculations across different CCPs.

It would be insightful to compare the Tranches and Ratio approaches to the Delta method-based approximation of expected uncapped loss of clearing member $i$,

$$E[L^{uc,s}_i] = E\left[\frac{DF_i}{DF_{s,i}} \left(\sum_{j \neq i} C_j Y_j - DF\right)^+\right] \approx \frac{DF_i}{DF - \left(\sum_{j \neq i} DF_j p_j\right)} E\left[\left(\sum_{j \neq i} C_j Y_j - DF\right)^+\right],$$

where $DF_{s,i} = DF - \sum_{j \neq i} DF_j Y_j$, $p_j$ is $CM_j$’s default probability at time $T$; $i = 1, ..., n$, and the CCP’s equity contribution, $E$, is assumed to be zero.

Appendix D’s normal copula-based approximation of clearing members’ expected uncapped loss for large $n$ gives a second model-based approximation of the CCP risk capital.\textsuperscript{24} For instance, consider Example 2 in Appendix D, where default indicators are exchangeable random variables that are correlated based on a single-factor normal copula threshold model. For large $n$ and in the absence of the CCP’s equity contribution, the $CM_i$’s expected uncapped loss can be approximated by,

$$E[L^{uc,s}_i] \approx \frac{DF_i}{DF(1 - p) + pDF_i} \left(\theta_1 \sum_{i=1}^{n} C_i + \theta_2 \left(\sum_{i=1}^{n} C_i^2\right)^{\frac{1}{2}}\right),$$

where $p$ is clearing members’ default probability. The coefficients $\theta_1$ and $\theta_2$ depend on $p$, the correlation between normal random variables underlying the copula model, and the confidence level, $\alpha$, associated with $DF$. For the numerical examples of Appendix D, the order of $\theta_1$ varies from $10^{-2}$ to $10^{-4}$, and the order of $\theta_2$ varies from $10^{-3}$ to $10^{-5}$.

6 Monte Carlo CCP Risk Measurement

Monte Carlo simulation is widely used by different types of financial institutions in market, credit, and counterparty credit risk measurement. Some of the well-established Monte Carlo schemes in risk management can be directly employed in the proposed CCP risk measurement

\textsuperscript{23}The risk weight in the Ratio approach is set equal to 12.5. Also, (21) is a simplified version of the Ratio approach. Similar to our comment in the previous footnote, we have not considered $RLDF = \max\{DF^{cover*}, K_{ccp}\}$ as it does not seem to have a sound conceptual foundation.

\textsuperscript{24}These approximate expected loss formulations have been presented for the comparison with the current regulatory CCP risk capital framework. As stated before, t-copula threshold models are preferable to normal copula models.
framework to estimate all layers of the default waterfall resources. We, first, provide a short summary on how a typical derivatives CCP estimates the initial margin of its clearing members. Next, we briefly discuss and outline Monte Carlo estimation of prefunded default fund contributions and the CCP risk capital of clearing members.

It is well-known from the risk management common practice and academic literature that Monte Carlo derivatives portfolio risk estimation involves two steps; first the portfolio’s underlying risk factors, e.g., equity prices, commodity prices, interest and exchange rates, are generated up to a given point on a discrete time grid. Then, conditional on the realization of the risk factors, portfolio constituents are valuated. As closed form derivatives pricing formulas are rarely available, the second step involves some approximations or additional layers of Monte Carlo simulation (see, e.g., Chapter 9 of Glasserman [2004], Broadie et al. [2011], Gordy and Juneja [2010], and the references there).²⁵

Initial Margin is the only model-based component of CCPs’ default waterfall resources. Derivatives CCPs’ (initial) margin models define a clearing member’s IM based on a particular risk measure, i.e., VaR or ES, associated with the clearing member’s derivatives portfolio. For instance, consider the equidistant time grid 0 ≡ t₀ < t₁ < ... < tₙ ≡ T with h = tᵢ - tᵢ₋₁; let V ≡ Vᵢ denote the positive part of the CCP’s derivatives portfolio value with CMᵢ. The VaR-based IM ≡ IMᵢ at time tᵢ estimated at time tᵢ₋₁ for a given confidence level ˜α and based on a fixed MPOR, h, is defined as,

\[ IMᵢ ≡ VaR_{˜α,tᵢ₋₁}(∆Vᵢ) = \inf\{r ∈ \mathcal{R} : P(∆Vᵢ > r | Fᵢ₋₁) ≤ 1 - ˜α\}, \]

where,

\[ ∆Vᵢ = V(tᵢ) - V(tᵢ₋₁), \]

and Fᵢ₋₁ denotes the information set (filtration) generated by the portfolio’s underlying risk factors by time tᵢ₋₁. Suppose that CMᵢ joins the CCP at time zero; its IM is set equal to IM₁ ≡ IM(t₁) and is estimated at time 0 by Monte Carlo. The CMᵢ’s time dependent IM is then updated sequentially based on the above mentioned time grid. For instance, after IM₁ is estimated and when the market information is revealed at time t₁, i.e., conditional on F₁, the CCP estimates IM₂.

### 6.1 Monte Carlo Estimation of the Prefunded Default Funds

As stated in Section 2.1, we suggest using expected shortfall at a confidence level α to define the total prefunded default fund, DF = E[L | L ≥ VaR_α(L)], where L = \sumᵢₙ Cᵢ Yᵢ. Recall (3). First, we discuss Monte Carlo estimation of EPE-based Cᵢ’s,

\[ EPEᵢ = \int_0^T E[eᵢ(t)] dt, \]

where eᵢ(t) denotes the CCP’s collateralized time-t exposure to CMᵢ at its default in the presence of variation an initial margin as defined in (2). Ghamami and Zhang [2013] introduce an efficient

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²⁵The first step is usually done using the physical measure, and the valuation step uses the risk neutral measure.
Monte Carlo framework for EPE, effective EPE and CVA estimation in the absence of initial margin that can be employed by CCPs. Monte Carlo estimation of EPE in the presence of initial margin will be computationally more demanding. To see this, recall (2), and set \( R_i(t + \Delta) - V_i^+(t - \tilde{\Delta}) \). Suppose that IM is defined based on \( \text{VaR}_{\alpha}(R_i) \equiv \text{VaR} \). Note that expected collateralized exposure can then be written as,

\[
E[e_i(t)] = E[(R_i - \text{VaR})^+] = E[R_i1\{R_i > \text{VaR}\}] - \hat{\alpha}\text{VaR}.
\]

Given the confidence level, \( \hat{\alpha} \), \( P(R_i > \text{VaR}) \), which is less than or equal to 1 - \( \hat{\alpha} \) is usually small, in which case, crude Monte Carlo will lead to expected exposure estimates with large variance. Successful application of standard variance reduction techniques, e.g., importance sampling for rare event simulation, is usually problem specific and not straightforward at derivatives portfolio level. It, then, becomes quite crucial that CCPs increase their computational budget when Monte Carlo is used for EPE or eEPE-based estimation of \( C_i \)'s.

After \( C_1, ..., C_n \) are estimated, a derivatives CCP uses the two-step procedure outlined in Section 2 to specify \( DF \) based on expected shortfall and then decomposes \( DF \) to \( DF_1, ..., DF_n \). That is, Monte Carlo is to be used to estimate,

\( q \equiv \text{VaR}_\alpha(L), DF \equiv E[L|L > q], \) and \( DF_i \equiv E[C_iY_i|L > q], \) \( i = 1, ..., n, \)

given a confidence level \( \alpha \in (0, 1) \). Estimation of \( DF \) and \( DF_i \)'s can be thought of as a two-phase procedure, where one first estimate VaR and then estimates the two conditional expectations above using the estimated VaR from the first phase in place of the true VaR, (see Section 9.2 of Glasserman [2004] and Brereton et al. [2013] for efficient Monte Carlo estimation of value at risk).

Since the loss probability, \( P(L > q) \), which is less than or equal to 1 - \( \alpha \), is typically small, crude Monte Carlo would require a large number of runs to achieve a satisfactory variance for the estimators of VaR and ES. Consider the ratio representation of expected shortfall,

\[
E[L|L > q] = \frac{E[L1\{L > q\}]}{P(L > q)}.
\]

A Monte Carlo scheme that works well for estimation of the loss probabilities, \( P(L > q) \), will often work well for estimating the numerator on the right side above because \( E[L1\{L > q\}] \) takes positive values only when the event \( \{L > q\} \) occurs.

Consider the single-factor equivalent of the \( t \)-copula threshold model given in display (1),

\[
Y_i = 1\{X_i > x_i\} \text{ and } X_i = \frac{a_iZ + \sqrt{1 - a_i^2\xi_i}}{\lambda}, \quad i = 1, ..., n.
\]

Note that when \( \lambda \) takes values close to zero, all the \( X_i \)'s are likely to be large, leading to many simultaneous defaults. Both empirically and using asymptotic approximations, Bassamboo et al. [2008] have illustrated when \( P(L > q) \) is small – of order \( 10^{-3} \) or less based on their numerical
examples – the event \( \{L > q\} \) happens primarily when \( \lambda \) takes small values while \( Z \) and \( \xi_i \)'s have little influence on the occurrence of large losses, (see Theorem 1 and Tables 1-5 of their paper). Bassamboo et al. have introduced an efficient exponential-twisting based importance sampling (IS) algorithm for estimating loss probabilities and expected shortfall. In words, their IS algorithm first generates \( Z \) based on its original distribution function and samples from \( \lambda \) based on its exponential-twisted distribution. Next, conditional on the Monte Carlo realizations in the previous step, the conditionally independent Bernoulli random variables, \( Y_1, ..., Y_n \), are sampled from according to their original mean, \( p_1, ..., p_n \), or they are being generated based on an exponentially-twisted IS probability mass function (see Section 4.1.1 and 4.2 of their paper). When loss probabilities are of order \( 10^{-3} \) or less the above mentioned two-step IS algorithm outperforms crude Monte Carlo significantly and can be used by CCPs for estimating prefunded default fund contributions.

6.2 Monte Carlo Estimation of the CCP Risk Capital

The Monte Carlo estimated prefunded default funds, \( DF, DF_1, ..., DF_n \), of the previous section are used in place of the true, \( DF, DF_1, ..., DF_n \), in Monte Carlo estimation of the CCP risk capital in this section. Assume that the unfunded default funds are capped and \( CM_i \)'s CCP risk capital is defined based on the total expected loss,

\[
E[L_t^{\sigma}] = E[L_t^i] + \bar{U}_i P_{ccp,i},
\]

as defined by (17) and (19). First, note that \( \bar{U}_i \), the \( CM_i \)'s loan-equivalent exposure to the CCP at its default, which is defined in (18), can be estimated with Monte Carlo efficiently using the framework introduced by Ghamami and Zhang [2013]. Consider the expected capped loss of \( CM_i \) assuming its time-\( T \) survival,

\[
E[L_t^i] = E \left[ \min \left\{ \frac{DF_i}{DF_{s,i}} \left( \sum_{j \neq i} C_j Y_j - E - DF \right)^+ , \beta DF_i \right\} \right].
\]

Note that \( L_t^i \) takes positive values only if,

\[
\sum_{j \neq i} C_j Y_j > E + DF.
\]

When the above mentioned event is a rare event, the importance sampling algorithm of Bassamboo et al. [2008], which places further probability mass on this event, can be used for efficient estimation of \( E[L_t^i] \). Now, consider \( P_{ccp,i} \), the CCP’s default probability from the \( CM_i \)'s perspective, i.e., assuming its time-\( T \) survival,

\[ ^{26} \text{Bassamboo et al. have also introduced a second importance sampling algorithm based on hazard rate twisting; their numerical examples indicate that the exponential-twisting algorithm outperforms the other for loss-probability estimation.} \]
\[ P_{ccp,i} = P \left( \sum_{j \neq i} C_j Y_j > E + DF + \tilde{DF}_i + \sum_{j \neq i} \tilde{DF}_j \right), \]

as derived in (26) of Appendix B. Note that \( P_{ccp,i} \leq P(\sum_{j \neq i} C_j Y_j > E + DF) \), and so the IS algorithm of Bassamboo et al. [2008] should work well for Monte Carlo estimation of \( P_{ccp,i} \) as it puts more probability mass on the event \( \{ \sum_{j \neq i} C_j Y_j > E + DF \} \). Display (26) of Appendix B gives the conservative upper bound, \( 1 - \alpha \), on \( P_{ccp,i} \). So, a conservative approximation of \( P_{ccp,i} \) can be also used based the confidence level \( \alpha \) associated with DF.

7 Conclusion

The opacity of the OTC markets has been widely considered as one of the major causes of the 2007-2009 financial crises, (see Part 4 of Archarya and Richardson [2009]). Derivatives CCPs are to make the OTC markets safer by reducing this lack of transparency following the 2009 G20 clearing mandate. International regulatory risk management standards have historically influenced – and to some extent shaped – the dynamics of financial markets. Before the clearing mandate, a detailed model-based focus merely on CCPs’ margin requirements may have served the financial system well. Post clearing mandate, however, it is crucial that the CCP risk would be managed based on a coherent framework across all the default waterfall resources. This is, in part, because CCP risk management rules and practices will impact the capital structure of clearing members through the regulatory CCP risk capital requirements; they will also affect the relative costs of central versus bilateral clearing. In the absence of a well defined CCP risk measurement framework, a non-unifiable central clearing environment may replace the opaque OTC markets.

The proposed framework assumes that a CCP’s clearing members may default at the end of a fixed time interval and approximates the collateralized exposures at default in the presence of only variation and initial margin. The remaining default waterfall resources, i.e., the prefunded and unfunded default funds, are then defined rigorously based on the credit loss distribution of the CCP’s portfolio of clearing members’ portfolios. Based on the proposed one-period model, the unanticipated potential losses of a clearing member can be expressed as the sum of the potential losses due to the CCP’s unfunded default fund capital calls resulting from the possible default of other members and the replacement cost that the clearing member may face if the CCP defaults. This replacement cost depends on the values of the derivatives portfolio that the clearing member holds with the CCP in a given time interval. The CCP risk capital of clearing members is then derived based on the average or quantile of the unanticipated potential losses. Since the default waterfall resources of derivatives CCPs are mathematically modeled, the CCP risk capital of clearing members can be specified rigorously. This model-based risk sensitive approach to the CCP risk capital can be used to develop and evaluate non-model-based approaches that are less risk sensitive.
Appendix

A The Allocation Rule for Unfunded Default Funds

Consider the total prefunded default funds based on expected shortfall at a given confidence level, i.e., $DF \equiv E[L|L > q]$, where $L = \sum_{i=1}^{n} C_i Y_i$ and $q \equiv \text{VaR}_\alpha(L)$, and the Euler-based decomposition of $DF$,

$$DF_i = C_i E[Y_i|L > q].$$

Consider the allocation of $L^{(2)} = (\sum_{i=1}^{n} C_i Y_i - E - DF)^+$ to $CM_i$ proportional to $DF_i$’s and $C_i$’s,

$$\frac{DF_i \bar{Y}_i}{\sum_{j=1}^{n} DF_j \bar{Y}_j} L^{(2)}, \quad \text{and} \quad \frac{C_i \bar{Y}_i}{\sum_{j=1}^{n} C_j \bar{Y}_j} L^{(2)},$$

where $\bar{Y}_j = 1 - Y_j, j = 1, ..., n$. In addition to the market risk represented by $C_i$, the $CM_i$’s prefunded default fund also depends on $Y_i$, i.e., the $CM_i$’s credit quality, and the correlation among clearing members through the event $\{L > q\}$, which makes $DF_i$’s more suitable than $C_i$’s for allocation of $L^{(2)}$. This risk management justification of the DF-based allocation of unfunded default funds can be also seen from the following stylized example.

Example 1 Consider a CCP with 3 clearing members where $C_i \equiv 1, i = 1, 2, 3$. Let $P_{k_1k_2k_3} = P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3); k_j = 0, 1, j = 1, 2, 3$, and suppose that the joint probability mass function of the default indicators is as follows,

$P_{000} = .64, P_{100} = .06, P_{010} = .06, P_{001} = .08, P_{110} = .01, P_{101} = .08, P_{011} = .03, P_{111} = .04,$

which gives the marginal default probabilities, $p_1 = .19, p_2 = .14, p_3 = .23$. Let

$$w_i = E[Y_i|Y_1 + Y_2 + Y_3 \geq 2], \quad i = 1, 2, 3.$$  

So, $w_1 = 13/16, w_2 = .5,$ and $w_3 = 15/16$. First, consider the $C$-based allocation rule using the right side of (22). Conditional on $CM_3$’s default and $CM_1$ and $CM_2$’s survival at time $T$, i.e., conditional on the event $\{Y_1 = 0, Y_2 = 0, Y_3 = 1\}$, the CCP’s potential loss will be distributed equally among $CM_1$ and $CM_2$ regardless of their credit quality and correlation. However, under the $DF$-based allocation rule given by the left side of (22), and conditional on $\{Y_1 = 0, Y_2 = 0, Y_3 = 1\}$, the CCP allocates $w_1/(w_1 + w_2) = 13/21$ of the potential loss to $CM_1$ and $8/21$ of it to $CM_2$. This shows that the $DF$-based allocation rule is more desirable since, conditional on $\{Y_1 = 0, Y_2 = 0, Y_3 = 1\}, CM_1$, whose credit quality is lower than $CM_2$, is allocated more of the CCP’s potential loss. Also, note that in addition to the marginal credit qualities, the $CM_1$’s credit quality conditional on the event $\{Y_1 + Y_2 + Y + 3 \geq 2\}$ is lower compared to that of $CM_2$, i.e., $w_1 > w_2$. 

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B The CCP’s Default Probability in the One-Period Model

Consider the CCP’s default probability in the one-period model. Let $A$ denote the event that the CCP defaults and $N_d$ denote the number of defaults at time $T$. Conditioning on the number of defaults at time $T$, we have,


(23)

Note that when the unfunded default funds are uncapped, $P(A|N_d < n)$ is zero, and so the CCP’s default probability in the uncapped case becomes,

$$P(\sum_{i=1}^{n} C_i > E + DF|N_d = n)P(N_d = n).$$  

(24)

When the unfunded default fund contributions are capped by a multiple of the clearing members’ prefunded default funds, $P(A|N_d < n)$ could be positive and is expressed as follows,

$$P(A|N_d < n) = P(L^{(2)} > \tilde{DF}|N_d < n) = P(\sum_{i=1}^{n} C_i Y_i > E + DF + \tilde{DF}|N_d < n),$$  

(25)

where $L^{(2)}$ is defined in (10), the capped unfunded default contribution of $CM_i$, denoted by $\tilde{DF}_i \equiv L_i$, is defined in (12), and $\tilde{DF} \equiv \sum_{i=1}^{n} \tilde{DF}_i$.

As stated in Section 2.2, it will be more conservative in CCP risk capital calculations to define a clearing member’s potential losses due to the possible default of other members assuming its survival at time $T$. Under this assumption we have, $P(N_d = n) = 0$ and $P(N_d < n) = 1$ in (23). That is, from $CM_i$’s perspective, $P_i(A) \equiv P(A|N_d < n)$. Consequently, in the uncapped case, a surviving member assigns zero to the central counterparty’s default probability. The CCP’s default probability in the capped case from the $CM_i$’s perspective assuming its survival at time $T$ is formulated as follows,

$$P_{ccp,i} \equiv P_i(A) = P \left( \sum_{j \neq i} U_j Y_j - E - DF_{s,i} > \hat{DF}_i^s + \sum_{j \neq i} \hat{DF}_j \right)$$

$$\leq 1 - \alpha,$$  

(26)

where $DF_{s,i}$, denoting the prefunded default fund of surviving members assuming $CM_i$’s survival, is defined in (14), and $\hat{DF}_i \equiv L_i$ and $\hat{DF}_i^s \equiv L_i^s$; $i = 1, \ldots, n$, are defined in (12) and (15), respectively. Note that, clearly, in formulating the CCP’s default probability from the $CM_i$’s perspective, $P_{ccp,i}$, other clearing members are not assumed to survive at time $T$. The last equality on the right side above is derived based on simple algebraic manipulations used to derive (10). The upper bound, $1 - \alpha$, on CCP’s default probability is derived by using our proposed specifications of the total prefunded default fund contributions given in (6)-(7).
C  Total Losses to the Clearing Members: The Unconditional Case

As illustrated in Appendix B, the CCP may default in the uncapped case only when all clearing members default at time $T$. So, when the unfunded default funds are uncapped, the total loss to $CM_i$ will be only due to the default of other clearing members. That is,

$$L_{t,uc}^i = L_{uc}^i,$$

where $L_{uc}^i$ is given by (11). Let $\tilde{Y} \equiv 1\{L(2) > \tilde{DF}\}$ denote the CCP’s default indicator when $CM_i$’s unfunded default fund is capped by a multiple of $DF_i$. Then, the total potential loss in the capped case becomes,

$$L_t^i = L_i + \tilde{U}_i(1 - Y_i)\tilde{Y},$$

where $E[\tilde{Y}]$ is given in (23). Also, $\tilde{U}_i$ denotes the $CM_i$’s loan-equivalent time-$T$ exposure at default to the CCP and is defined in (18).

D  Expected Loss Approximations for Large CCPs Based on Normal Copulas

Consider the CCP risk capital of clearing member $i$ based on expected uncapped loss in the absence of the CCP’s equity contribution,

$$E[L_{i,uc,s}^i] = E \left[ \frac{DF_i}{DF_{s,i}} \left( \sum_{j \neq i} C_j Y_j - DF \right)^+ \right],$$

(27)

where $DF_{s,i} = DF - \sum_{j \neq i} DF_j Y_j$. Suppose that the the dependence among $Y_i$s is defined based on a one-factor normal copula threshold model as follows,

$$Y_i = 1\{X_i < x_i\} \text{ and } X_i = \rho_i Z + \sqrt{1 - \rho_i^2} \xi_i, \quad i = 1, ..., n,$$

where $Z, \xi_1, ..., \xi_n$ are independent standard normal random variables and $E[Y_i] = p_i$, (setting $\rho_i \equiv \rho$, the above mentioned normal copula threshold model becomes the Vasecek [1987] single-factor loan portfolio model). Note that conditional on $Z$, default indicators are independent with mean

$$E[Y_i|Z] = \Phi \left( \frac{\Phi^{-1}(p_i) - \rho_i Z}{\sqrt{1 - \rho_i^2}} \right) = \Phi_{i,z}.$$
where \( \Phi \) denote the normal distribution function.

When the number of clearing members, \( n \), is sufficiently large, we approximate \( L = \sum_{i=1}^{n} C_i Y_i \) conditional on \( Z \), with a normal random variable with mean \( \mu_z \) and variance \( \sigma_z^2 \) specified as follows,

\[
\mu_z \equiv \sum_{i=1}^{n} C_i \Phi_{i,z}, \quad \text{and} \quad \sigma_z^2 \equiv \sum_{i=1}^{n} C_i^2 \Phi_{i,z}(1 - \Phi_{i,z}).
\]

This approximation is based on the central limit theorem (CLT) for independent random variables. More specifically, let \( L_z \equiv L \mid Z \), for any given value of \( Z \), the normalized random variable, \( \frac{L_z - \mu_z}{\sigma_z} \), converges in distribution to a standard normal random variable as \( n \to \infty \) if \( C_i \)'s are uniformly bounded, i.e., for some \( M \), \( C_i < M \) for all \( i \), and if for any given value of \( Z \), \( \sigma_z^2 \to \infty \) as \( n \to \infty \), (see, e.g., Chapter 2 of Durrett [2005], the Lindeberg-Feller theorem). \(^{27}\)

Consider the VaR based specification of the prefunded default funds, i.e., \( DF = \text{VaR}_\alpha(L) \equiv \text{VaR}_\alpha \). Suppose that \( n \) is sufficiently large and replace \( \sum_{j \neq i} C_j Y_j \) with \( \sum_{i=1}^{n} C_i Y_i \). Conditioning on \( Z \), the expected value of the second term on the right side of (27) can be written as,

\[
E \left[ \frac{L_z - \mu_z}{\sigma_z} \right] = E \left[ E \left[ L \mid \{L > \text{VaR}_\alpha \} \right] \mid Z \right] - (1 - \alpha) \text{VaR}_\alpha.
\]

Let \( \phi \) denote the density of the standard normal. Consider the first term on the right side above. Note that when \( L_z \equiv L \mid Z \sim N(\mu_z, \sigma_z^2) \). So,

\[
E[L_z \mathbf{1}\{L_z \geq \text{VaR}_\alpha(L_z)\}] = \mu_z + \sigma_z \phi(\Phi^{-1}(\alpha)).
\]

The above formula is derived easily, (see, e.g., Chapter 2 of McNeil et al. [2005]). Also, we use Proposition 8.16 of McNeil et al. [2005] (or Propositions 3 and 4 of Gordy [2003]) to approximate \( \text{VaR}_\alpha(L) \) as follows,

\[
\text{VaR}_\alpha(L) \approx \sum_{i=1}^{n} C_i \Phi \left( \frac{\Phi^{-1}(p_i) - \rho_i \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_i^2}} \right) \equiv \hat{\text{VaR}}_\alpha(L).
\]

So, we can write,

\[
E \left[ (L - \text{VaR}_\alpha)^+ \right] \approx E[\mu_z] + \phi(\Phi^{-1}(\alpha))E[\sigma_z] - (1 - \alpha) \hat{\text{VaR}}_\alpha(L).
\]

\(^{27}\)In words, if \( n \) is sufficiently large and if \( C \)'s are not dominated by a few \( C_i \)'s much larger than the rest, the CLT holds, and the normal approximation of \( L_z \) would be a good approximation.
Using the Delta method to approximate the expected value of the ratio on the right side of (27) with ratio of the expectations, the approximate expected uncapped loss of clearing members for large $n$ becomes,

$$
E[L_{i}^{uc,s}] \approx \frac{DF_{i}}{DF - \sum_{j=2}^{n} DF_{j} p_{j}} \left( E[\mu_{z}] + \phi(\Phi^{-1}(\alpha)) E[\sigma_{z}] - (1 - \alpha) \hat{VaR}_{\alpha}(L) \right), \tag{29}
$$

where the expectation on the right side can be easily calculated using a numerical method.

**Example 2: The Exchangeable Case**  Consider the single-factor normal copula threshold model above with $\rho_{i} = \rho$ for $i = 1, ..., n$, which makes default indicators $Y_{1}, ..., Y_{n}$ exchangeable Bernoulli random variables with $E[Y_{i}] = p$. When $n$ is sufficiently large, $L$ conditional on $Z$ can be approximated by a normal random variable, and the expected uncapped loss approximation in the exchangeable case becomes,

$$
E[L_{i}^{uc,s}] \approx \frac{DF_{i}}{DF(1 - p) + pDF_{i}} \left( \theta_{1} \sum_{i=1}^{n} C_{i} + \theta_{2} \left( \sum_{i=1}^{n} C_{i}^{2} \right)^{\frac{1}{2}} \right), \tag{30}
$$

where $\theta_{1} \equiv E[\Phi_{z}] - (1 - \alpha) \Phi_{v}$ and $\theta_{2} \equiv \phi(\Phi^{-1}(\alpha)) E[\Phi_{z}^{\frac{1}{2}} (1 - \Phi_{z})^{\frac{1}{2}}]$, with $\Phi_{z} = \Phi(\frac{\Phi^{-1}(p) - \rho Z}{\sqrt{1 - \rho^{2}}})$, and $\Phi_{v} = \Phi(\frac{\Phi^{-1}(p) - \rho \Phi^{-1}(\alpha)}{\sqrt{1 - \rho^{2}}})$. Table 1 and Table 2 give estimations for $\theta_{1}$ and $\theta_{2}$ for different model parameters; $\rho = .05, .9, \alpha = .99, .9975$, and $p = .01, .001$. The average-based uncapped loss formula (30) with simple numerical examples of Tables 1 and 2 can provide an approximate way of looking at a clearing members’ CCP risk capital for comparison with the current regulatory framework discussed in Section 5.

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Table 1: $\alpha = .99$

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Table 2: $\alpha = .9975$

**References**


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